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# Real Analysis I

Unit I

Functions of Bounded variation

Introduction - properties of monotonic functions - Functions of bounded variation - Total variation - Additive Property of total variation - Total variation on [a,x] as a function of x - Functions of bounded variation expressed as the difference of two increasing functions - continious functions of bounded variation

Chapter 6 : Sec 6.1 to 6.8

Unit II

The Riemann - Stieltjes Integral

Introduction Notation - the def of the Riemann - Stieltjes Integral - Linear properties - Integration by parts - change of variable in a Riemann - Stieltjes Integral - Reduction to a Riemann Integral - Euler's summation formula - monotonically increasing integrators - upper and lower integrals - Additive and linearity properties of upper and lower integrals - Riemann's conditions

Chapter 7 : Sec 7.1 to 7.13

Unit III Continue

Integrators of bounded variation - Sufficient conditions for the existence of Riemann - Stieltjes integrals - mean value theorems for Riemann - Stieltjes integrals - the Integrals as a function of the Interval - Second fundamental theorem of Integral Calculus - change of variable in a Riemann integral

Second mean value theorem for Riemann integral - Riemann stieltjes integrals depending on a parameter - Differentiation under the integral sign

Chapter 7 : Sec 7.15 to 7.25

Unit IV

Infinite series and Infinite products:

Absolute and conditional convergence - Dirichlet's test and Abel's test - Rearrangement of series - Riemann's theorem on conditionally convergent series. Double sequences - Double series - Rearrangement theorem for double series - A sufficient condition for equality of iterated series - Multiplication of series Cesaro summability - Infinite products

chap 8 : sec 8.8, 8.15, 8.17, 8.18, 8.20, 8.21 to 8.26

Unit V

Sequences of functions:

pointwise convergence of sequences of functions - Examples of sequences of real valued functions - Definition of uniform convergence - uniform convergence and continuity - the cauchy condition for uniform convergence - uniform convergence of Infinite series of functions - uniform convergence and Riemann-stieltjes integration - uniform convergence and Differentiation - sufficient condition for uniform convergence of a series - mean convergence.

Chn 9 : Sec 9.1 to 9.6, 9.8, 9.10, 9.11, 9.13.

Recommended Text :

Tom M. Apostal : Mathematical Analysis - 2nd edition Addition - Wesley publishs company Inc, New year 1997.

Book :

1. Goutke R.G. Real analysis 1976
2. Rudin, W. principles of mathematical Analysis, New Year 1976
3. malik, S.C savita Arole, mathematical Analysis New Delhi 1991

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Def: Real valued function:

When $f : A \to R$ we call 'f' a real valued function.

Def

If 'f' is a real valued function on an interval I of R such that $I \subset R$. We say that 'f' is Increasing (or) Non Decreasing on I, if

$f(x) \leq f(y)$, $\forall\ x, y \in I$ and $x < y$

Def: Decreasing (or) Non-Increasing function

If 'f' is a real valued function on an interval I of R such that $I \subset R$, We say the that 'f' is Decreasing (or) Non Increasing on I, if

$f(x) \geq f(y)$, $\forall\ x, y \in I$ and $x > y$

Def: Strictly Increasing function:

If f is a real valued function on an Interval I. we say f is strictly Increasing on I, if.

$f(x) < f(y)$, $\forall\ x, y \in I$. and $x < y$

Def: Strictly Decreasing function:

If f is a real valued function on an Interval I. we say 'f' is strictly Decreasing on I if,

$f(x) > f(y)$, $\forall\ x, y \in I$ and $x > y$

Monotonic function:

If $f$ is real valued function on $I$ then $f$ is called a monotonic function if $f$ is either non decreasing (or) non-Increasing on $I$.

Def: Right hand limit:

If '$f$' is a real valued function on $[a,b]$ assume $c \in [a,b]$. if $f(x) \rightarrow A$ as $x \rightarrow c$ through values greater than $c$, we say that 'A' is the 'RIGHT HAND LIMIT' of $f$ at $c$ and we indicate by

$$\lim_{x \to c+} f(x) = A = f(c+)$$

the Right hand limit of $f$ at $c$ is $A$ and is denoted by $A = f(c+)$ then for any given $\epsilon > 0$ we can find corresponding $\delta > 0$ such that ~~$|f(x) - f(c+)|$~~

$$|f(x) - f(c+)| < \epsilon \text{ whenever } c < x < c+\delta < b$$

Note that '$f$' need not be defined at the point '$c$' itself.

If '$f$' is defined at $c$ and if $f(c+) = f(c)$ we say that '$f$' is continuous from the right at '$c$'.

Left hand limit at continuous from the left at $c$

If $c$ in $[a,b]$ $c \in [a,b]$

If $a < c < b$ then $f$ is continuous at $c$

$$c \Leftrightarrow f(c+) = f(c-) = f(c)$$

Discontinuous :

we say that $c$ is a discontinuity of $f$ if $f$ is not continuous at $c$. if $c \in [a,b]$, if $a \leq c \leq b$ then $f$ is discontinuous, we have

(i) Either $f(c+)$ (or) $f(c-)$ does not exist

(ii) Both $f(c+)$ and $f(c-)$ exist but different values

(iii) Both $f(c+)$ and $f(c-)$ exists and .

$$f(c+) = f(c-) \neq f(c)$$

Let '$f$' be defined on $[a,b]$. If $f(c+)$ and $f(c-)$ both exists at some interior paint '$c$', then

i) $f(c) - f(c-)$ is called left hand Jump of '$f$' at '$c$'

(ii) $f(c+) - f(c)$ is called Right hand Jump of '$f$' at '$c$'

(iii) $f(c+) - f(c-)$ is called Jump of '$f$' at '$c$'

. If any one of these . 3 numbers is different from zero, the '$c$' is called a Jump Discontinuity of '$f$'

Theorm ①

If $f$ is increasing on $[a,b]$ then $f(c+)$ and $f(c-)$ both exists for each $c'$ in $(a,b)$ and we have.

$$f(c-) \leq f(c) \leq f(c+)$$

At the end paints, we have

$$f(a) \leq f(a+) \text{ and } f(b-) \leq f(b)$$

Theorem ②

Let 'f' strictly increasing on $S$ in $R$ then $f^{-1}$ exists and is strictly increasing on $S$.

Theorem ③

Let 'f' be strictly increasing and continuous on compact interval $[a,b]$ then $f^{-1}$ is continuous and strictly increasing on the Interval $[a,b]$

Theo properties of Monotonic function

Theorem: 4

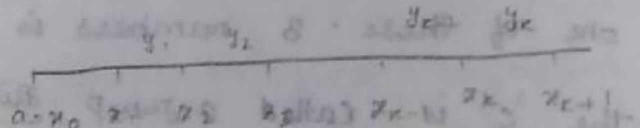
Let 'f' be an increasing function defined on $[a,b]$ and let $x_0, x_1, \dots x_n$ be $n+1$ points such that

$a = x_0 < x_1 < x_2 \dots < x_n = b$

then we have the inequality,

$$\sum_{k=1}^{n-1} \{ f(x_k^+) - f(x_k^-) \} \leq f(b) - f(a)$$

proof:



Given that $f$ is an increasing function defined on $[a,b]$ then at each point

$x_k \in (a,b)$ , $k = 1, 2, 3 \dots n-1$

we have the right limit

$f(x_k^+) = \lim_{x \to x_k^+} f(x)$ exists and also left hand limit

$f(x_k^-) = \lim_{x \to x_k^-} f(x)$ exists [By thm ①]

Assume $y_k \in (x_k, x_{k+1})$, $k = 1, 2, \ldots, n-1$

we have

$$x_k^+ \le y_k \ \& \ y_{k-1} \le x_k^-$$

$$f(x_k^+) \le f(y_k), \ f(y_{k-1}) \le f(x_k^-)$$

$\Rightarrow f(x_k^+) \le f(y_k)$ and $-f(y_{k-1}) \ge -f(x_k^-)$

$\Rightarrow f(x_k^+) - f(x_k^-) \le f(y_k) - f(y_{k-1})$, $k = 1, 2, \ldots, n-1$

$k = 1, 2, \ldots, n-1$

$f(x_1+) - f(x_1-) \le f(y_1) - f(y_0)$

$f(x_2+) - f(x_2-) \le f(y_2) - f(y_1)$

$f(x_3+) - f(x_3^-) \le f(y_3) - f(y_2)$

$\vdots$

$f(x_{n-1}^+) - f(x_{n-1}^-) \le f(y_{n-1}) - f(y_{n-2})$

$$\sum_{k=1}^{n-1} \{ f(x_k^+) - f(x_k^-) \} \le f(y_{n-1}) - f(y_0)$$

$\le .$

clearly $y_{n-1} < x_n$

$$f(y_{n-1}) \le f(x_n) = f(b)$$

$$a = x_0 < y_0$$

$$\Rightarrow f(x_0) = f(a) < f(b)$$

$$f(x_0) < f(y_0)$$

$$-f(x_0) \ge -f(y_0)$$

$$-f(a) \ge -f(y_0)$$

$$\therefore \sum_{k=1}^{n-1} \{ f(x_k+) - f(x_k-) \} \le f(b) - f(a)$$

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Theorem 5:

Q. P.T If $f$ is monotonic on $[a,b]$, then the set of discontinuities of $f$ is countable

(or)

S.T the set of discontinuities of a monotonically increasing function $f$ on $[a,b]$ is almost countable

Proof:

Assume that $f$ increasing on $[a,b]$

for $m>0$, let $x_1, x_2 \dots x_{n-1}$ be the points of $S_m$ the set of points in $(a,b)$ at which $f$ is discontinuities, such that

$$x_1 < x_2 < \dots < x_{n-1}$$

$$S_m = \{x_k \in (a,b) \,/\, f(x_k^+) - f(x_k^-) \geq \tfrac{1}{m}\}$$

where $m>0$, $k = 1, 2, \dots n-1$

P.T: $S_m$ is countable

It is enough to P.T $S_m$ is a finite set

ie) we shall P.T : $(n-1)$ is finite

By thm ④

$$\sum_{k=1}^{n-1} [f(x_k^+) - f(x_k^-)] \leq f(b) - f(a)$$

$$[f(x_1^+) - f(x_1^-)] + [f(x_2^+) - f(x_2^-)]$$

$$+ \dots + [f(x_{n-1}^+) - f(x_{n-1}^-)] \leq f(b) - f(a)$$

$$\frac{1}{m} + \frac{1}{m} \dots \frac{1}{m} \leq f(b) - f(a)$$

$(n-1)$ times

$$\frac{n-1}{m} \leq f(b) - f(a)$$

$n-1 \leq m f(b) - f(a)$

$\therefore n-1$ is finite

Thus $S_m$ is a finite set

$\therefore$ $S_m$ is a countable

But the set of all discontinuities of $f$ in $(a,b)$

is $\bigcup_{m=1}^{\infty} S_m$

$\therefore$ Each $S_m$ is countable and hence

"countable union of countable set is countable"

$\therefore \bigcup_{m=1}^{\infty} S_m$ is also countable

Thus the set of discontinuities of $f$ is countable,

if $f$ increasing on $[a,b]$.

Note: If $f$ is decreasing, the argument can be applied to $-f$, because $-f$ is increasing.

Functions of Bounded Variation:

Defn: Partition of $[a,b]$

If $[a,b]$ is a compact Interval, a set of points $P = \{x_0, x_1 \ldots x_n\}$ satisfying the Inequalities

$a = x_0 < x_1 < x_2 \ldots < x_n = b$ is called a partition of $[a,b]$.

The interval $[x_{k-1}, x_k]$ is called the $k^{th}$ subinterval of $P$ and we write,

$$\Delta x_k = x_k - x_{k-1}$$

So that $\sum_{k=1}^{n} \Delta x_k = x_n - x_0 = b-a$